

A CLASSIFICATION OF TETRAVALENT EDGE-TRANSITIVE METACIRCULANTS OF ODD ORDER

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ABSTRACT. In this paper a classification of tetravalent edge-transitive metacirculants is given. It is shown that a tetravalent edge-transitive metacirculant Γ is a normal graph except for four known graphs. If further, Γ is a Cayley graph of a non-abelian metacyclic group, then Γ is half-transitive.

1. INTRODUCTION

Throughout this paper graphs are assumed to be finite, simple, connected and undirected, unless stated otherwise.

A group is called *metacyclic* if and only if there exists a cyclic normal subgroup K of G such that G/K is cyclic. A metacyclic group is called *split* if it is a split extension of a cyclic group by a cyclic group, namely, $G = H:K$ is a semidirect product of a cyclic subgroup H by a cyclic subgroup K . A graph is called a *metacirculant* if it contains a metacyclic vertex-transitive automorphism group, introduced by Marušič and Šparl [31]. In this paper, a Cayley graph of a split metacyclic group G is called a *split metacirculant* of G .

A graph $\Gamma = (V, E)$ is called *X-edge-transitive* if a subgroup $X \leq \text{Aut}\Gamma$ is transitive on E . An *arc* of a graph is an ordered pair of adjacent vertices. A graph $\Gamma = (V, E)$ is called *X-half-transitive* if $X \leq \text{Aut}\Gamma$ is transitive on both V and E , but not transitive on the arc set of Γ ; in particular, if $X = \text{Aut}\Gamma$ then Γ is simply called a *half-transitive graph*.

In the literature, the class of metacirculants provides a rich source of various research projects, see for instance [2, 32, 36, 35, 41]. It has been therefore studied extensively in the past 30 years. Special classes of metacirculants have been well investigated, see [1, 9, 15, 17, 23, 26, 28] for edge-transitive Cayley graphs of special metacyclic groups; [24, 43] for half-transitive metacirculants of special order; [25] for cubic metacirculants; [31, 39, 40] for tetravalent half-transitive metacirculants. However, despite all the efforts there are still numerous questions about these graphs that need to be answered. Indeed, even the family of tetravalent half-transitive metacirculants, which has the smallest admissible valency for a half-transitive graph, seems to be too difficult to be completely classified. In this paper, we study tetravalent edge-transitive metacirculants of odd order and give a complete classification of these graphs.

A graph Γ is a *Cayley graph* if there exists a group G and a subset $S \subset G$ with $S = S^{-1} := \{s^{-1} \mid s \in S\}$ and $1 \notin S$ such that the vertices of Γ may be identified with the elements of G in the way that x is adjacent to y if and only if $yx^{-1} \in S$,

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such a graph is denoted by $\text{Cay}(G, S)$. A Cayley graph $\Gamma = \text{Cay}(G, S)$ has an automorphism group

$$\hat{G} = \{\hat{g} : x \mapsto xg, \text{ for all } x \in G \mid g \in G\},$$

consisting of right multiplications by elements $g \in G$. If further \hat{G} is normal in $\text{Aut}\Gamma$, then Γ is called a *normal Cayley graph*.

The main result of this paper is stated in the following theorem.

thm-1

Theorem 1.1. *Let Γ be a connected edge-transitive tetravalent metacirculant. Then Γ is split metacirculant of G and one of the following holds:*

- 1) Γ is an arc-transitive normal Cayley graph of an abelian group such that $\Gamma \neq K_5$;
- 2) Γ is a half-transitive normal Cayley graph of a non-abelian split metacyclic group G where $G \neq \mathbb{Z}_7:\mathbb{Z}_3, \mathbb{Z}_{11}:\mathbb{Z}_5$ or $\mathbb{Z}_{23}:\mathbb{Z}_{11}$. The automorphism group $\text{Aut}\Gamma = G:\mathbb{Z}_2$.
- 3) $\Gamma = \text{Cay}(G, S)$ with $G = \mathbb{Z}_5, \mathbb{Z}_7:\mathbb{Z}_3, \mathbb{Z}_{11}:\mathbb{Z}_5$ or $\mathbb{Z}_{23}:\mathbb{Z}_{11}$ and Γ is one of n non-isomorphic edge-transitive graphs. Among them, one is s -arc-transitive with $(\text{Aut}\Gamma, (\text{Aut}\Gamma)_\alpha, s)$ listed in Table 1, and the other $n - 1$ are half-transitive with $\text{Aut}\Gamma = G:\mathbb{Z}_2$.

$\text{Aut}\Gamma$	$(\text{Aut}\Gamma)_\alpha$	s	n	G
S_5	S_4	2	1	\mathbb{Z}_5
$\text{PGL}(2, 7)$	D_{16}	1	3	$\mathbb{Z}_7:\mathbb{Z}_3$
$\text{PGL}(2, 11)$	S_4	2	6	$\mathbb{Z}_{11}:\mathbb{Z}_5$
$\text{PSL}(2, 23)$	S_4	2	11	$\mathbb{Z}_{23}:\mathbb{Z}_{11}$

Table 1

A metacirculant Γ is called a *Sylow-circulant* if $\text{Aut}\Gamma$ has a transitive metacyclic subgroup of which all Sylow subgroups are cyclic.

Hypothesis (*): Let G be a split metacyclic group such that all Sylow subgroups of G are cyclic. Then G can be written as

$$G = H \times K = \langle c \rangle \times \langle a, b \rangle = \langle a \rangle : \langle bc \rangle,$$

where

$$K = \langle a, b \mid a^m = b^n = 1, a^b = a^r, r^n \equiv 1 \pmod{m} \rangle,$$

such that no Sylow p -subgroup of $\langle b \rangle$ lies in the centre of G . Let n_0 be the smallest integer such that $\langle b^{n_0} \rangle \in \mathbf{Z}(G)$.

We remark that H may be a trivial subgroup of G . In this case, c is the identity of G . Recall that the Euler Phi-function $\phi(m)$ is the number of positive integers that are less than m and coprime to m .

thm-2

Theorem 1.2. *Let $G = H \times K = \langle c \rangle \times \langle a, b \rangle$ and n_0 be as hypothesis (*). Let $\Gamma = \text{Cay}(G, S)$ be a connected tetravalent edge-transitive graph. Then*

$$S = \{cb^j, c^{-1}ab^j, c^{-1}b^{-j}, c(ab^j)^{-1}\},$$

where $1 \leq j < n_0$ and $(j, n) = 1$.

In particular, there are exactly $\phi(n_0)/2$ non-isomorphic tetravalent edge-transitive Cayley graphs of G , except for $G = \mathbb{Z}_{11}:\mathbb{Z}_5$ and $\mathbb{Z}_{23}:\mathbb{Z}_{11}$ which has 3 and 6 non-isomorphic tetravalent edge-transitive Cayley graphs of G , respectively.

2. NOTATION AND PRELIMINARIES

Let $\Gamma = \text{Cay}(G, S)$ be a Cayley graph of a group G . Then $\text{Aut}\Gamma$ contains the a regular subgroup which is isomorphic to G . For an automorphism group X such that $G \leq X \leq \text{Aut}\Gamma$, we have $X = GX_\alpha$, where $\alpha \in V$. Conversely, by [7, Proposition 4.3], if $\text{Aut}\Gamma$ has a regular subgroup G , then $\Gamma \cong \text{Cay}(G, S)$ for some $S \subset G$.

Besides the regular automorphism subgroup G , $\text{Aut}\Gamma$ has another important subgroup:

$$\text{Aut}(G, S) = \{\sigma \in \text{Aut}(G) \mid S^\sigma = S\}.$$

Let α be the vertex of Γ corresponding to the identity of G . Then $S = \Gamma(\alpha)$. $\text{Aut}(G, S) < \text{Aut}\Gamma$ and $\text{Aut}(G, S)$ fixes α and normalises G . Moreover, by [10, Lemma 2.1], we have the following lemma.

N(G) **Lemma 2.1.** *The normalizer $N_{\text{Aut}\Gamma}(G) = G:\text{Aut}(G, S)$.*

For any automorphism $\sigma \in \text{Aut}(G)$, the Cayley graphs $\text{Cay}(G, S)$ and $\text{Cay}(G, S^\sigma)$ are isomorphic. A Cayley graph $\text{Cay}(G, S)$ is called a *CI-graph* (CI stands for *Cayley Isomorphism*) if, whenever $\text{Cay}(G, T) \cong \text{Cay}(G, S)$, we have $T = S^\sigma$ for some automorphism $\sigma \in \text{Aut}(G)$.

CI-criterion

Proposition 2.2. ([16, Theorem 4.1]) *Let Γ be a Cayley graph of a group G , and let $A = \text{Aut}\Gamma$. If G is of odd order and $(|G|, |A_\alpha|) = 1$, then Γ is a CI-graph of G .*

Let G be a transitive permutation group on Ω . Then the G -orbits in $\Omega \times \Omega$ are called the G -orbits on Ω . For each orbital Δ , there is a *paired orbital* Δ^* , namely $\Delta^* := \{(\beta, \alpha) \mid (\alpha, \beta) \in \Delta\}$. If $\Delta = \Delta^*$, we say that the orbital Δ is self-paired.

For a G -orbital $\Delta := (\alpha, \beta)^G$, there is a di-graph Γ with vertex set Ω , and arc set $(\alpha, \beta)^G$. Γ is called an *orbital graph* of G on Ω . An orbital graph is undirected if and only if Δ is self-paired.

Let $\Gamma = (V, E)$ be a connected X -edge-transitive graph. Let N be an intransitive normal subgroup of X . Let \mathcal{B} be the set of N -orbits on V , which is sometimes denoted by V_N . The *quotient graph* $\Gamma_{\mathcal{B}}$ is the graph with vertex set \mathcal{B} such that two vertices $B, B' \in \mathcal{B}$ are adjacent in $\Gamma_{\mathcal{B}}$ if and only if there exist $\alpha \in B$ and $\alpha' \in B'$ with $\{\alpha, \alpha'\} \in E$. The graph $\Gamma_{\mathcal{B}}$ is sometimes denoted by Γ_N , called the *normal quotient* of Γ induced by N . Let K be the kernel of X acting on \mathcal{B} . Then $X/K \lesssim \text{Aut}\Gamma_N$ and X/K is edge-transitive on Γ_N .

As usual, for a set π of primes, denote by π' the set of primes which are not in π . For a group L , a *Hall π -subgroup* H of L is a subgroup such that each prime divisor of $|H|$ lies in π and the index $|L : H|$ is coprime to all primes in π . Let n be an integer, we denote n_π the divisor of n such that each prime divisor of n_π lies in π and n/n_π is coprime to n_π .

Hall

Lemma 2.3. *Let X be an insoluble group which has Hall π -subgroups. Then each normal or subnormal subgroup of X has Hall π -subgroups.*

Proof. Let $T \triangleleft M \triangleleft X$. Since $M \triangleleft X$, we have MX_π is a subgroup of X . So

$$|X| \geq |MX_\pi| = \frac{|X_\pi||M|}{|M \cap X_\pi|}.$$

Hence $|X|_\pi \geq |X|_\pi \cdot \frac{|M|_\pi}{|M \cap X_\pi|}$. It follows that $|M|_\pi = |M \cap X_\pi|$. So M has Hall π -subgroups. Similarly, T has Hall- π subgroups. \square

Let G be a metacyclic group. Then any subgroup or quotient group of G is a metacyclic group. The following lemma shows that G can not be written as a direct product of two isomorphic non-cyclic groups.

Lemma 2.4. *Let $G = G_1 \times G_2$ be a metacyclic group such that $G_1 \cong G_2$. Then $G_1 \cong G_2$ are cyclic groups*

Proof. Suppose $G_1 = H_1.K_1 \cong G_2 = H_2.K_2$, and $H.K = G = G_1 \times G_2$ be a metacyclic group, where $H_1 \cong H_2$, $K_1 \cong K_2$ and H, K are cyclic groups. We first assume that $G_1 \cong G_2$ are non-abelian groups. Since $H_i \times \mathbf{C}_{K_i}(H_i) \triangleleft G$ ($i = 1, 2$), we have $G_1/(H_1 \times \mathbf{C}_{K_1}(H_1)) \times G_2/(H_2 \times \mathbf{C}_{K_2}(H_2)) \cong (G_1 \times G_2)/(H_1 \times \mathbf{C}_{K_1}(H_1)) \times (H_1 \times \mathbf{C}_{K_1}(H_1))$. Since G_i are not abelian, there is a prime p such that the Sylow p -subgroup $(K_i)_p$ of K_i is not normal in G_i with $i = 1, 2$. So p divides $G_i/(H_i \times \mathbf{C}_{K_i}(H_i))$. Thus $\mathbb{Z}_p^2 \lesssim G_1/(H_1 \times \mathbf{C}_{K_1}(H_1)) \times G_2/(H_2 \times \mathbf{C}_{K_2}(H_2))$ is not cyclic. On the other hand, each normal subgroup of G is contained in $H_1 \times \mathbf{C}_{K_1}(H_1) \times H_2 \times \mathbf{C}_{K_2}(H_2)$, so does H . Thus $G/(H_1 \times \mathbf{C}_{K_1}(H_1)) \times (H_1 \times \mathbf{C}_{K_1}(H_1)) \lesssim G/H$ which is not possible as G/H is cyclic. So G_i ($i = 1, 2$) are abelian groups.

Now suppose $G_1 \cong G_2$ are non-cyclic abelian groups. Then there is a prime p such that $p \mid (|H_1|, |K_1|)$. So $\mathbb{Z}_p^4 \leq G$ which is not possible as G is a metacyclic group. So G_i ($i = 1, 2$) are cyclic groups. \square

Lemma 2.5 (Kazarin [14]). *Let T be a non-abelian simple group which has a Hall $2'$ -subgroup. Then $T = \text{PSL}(2, p)$, where $p = 2^e - 1$ is a prime. Further, $T = GH$, where $G = \mathbb{Z}_p : \mathbb{Z}_{\frac{p-1}{2}}$ and $H = D_{p+1} = D_{2^e}$.*

3. AUTOMORPHISMS OF EDGE-TRANSITIVE METACIRCULANTS

In this section, we study automorphism groups of tetravalent edge-transitive metacirculants. We first give a simple lemma on automorphism groups of tetravalent graphs of odd order.

Lemma 3.1. *Let Γ be a tetravalent graph of odd order. Let $X \leq \text{Aut} \Gamma$ be vertex-transitive on Γ . Suppose further that X is of odd order. Then X is regular.*

Proof. Let α be a vertex. Suppose that $X_\alpha \neq 1$. Then X_α is a 3-group since $|G|$ is odd and $|\Gamma(\alpha)| = 4$, and X_α divides $\Gamma(\alpha)$ into two orbits, one of which has length 3, and the other has length 1. Since Γ is undirected graph, we have Γ has an orbital graph Σ which is of valency 1. This is not possible, as Σ is a graph of odd order. Hence $X_\alpha = 1$, and X is regular. \square

Lemma 3.2. *Let $\Gamma = (V, E)$ be a connected tetravalent metacirculant of odd order. Then $\Gamma = \text{Cay}(G, S)$ for some metacyclic group G . Let $X \leq \text{Aut} \Gamma$ be such that $G \leq X$. Then each normal subgroup of X of odd order is a subgroup of G .*

Proof. By the definition of metacirculant, $\text{Aut } \Gamma$ has transitive metacyclic subgroups. Let G be such a subgroup of smallest order. We claim that G is of odd order. Suppose $|G|$ is even. Since G is metacyclic, we have G is soluble. Then a Hall $2'$ -subgroup $G_{2'}$ is of odd order. Since G is transitive, $X = GX_\alpha$. Moreover, $G_2 \leq X_\alpha$. Thus $X = G_{2'}X_\alpha$, so $G_{2'}$ is transitive on V . Now $G_{2'} < G$ which is contradict to our assumption. So G is of odd order. By Lemma 3.1, G is regular, so $\Gamma = \text{Cay}(G, S)$.

Let M be a normal subgroup of X of odd order. Let $Y := MG$. Then Y is transitive on $V\Gamma$, and $|Y|$ is odd order. By Lemma 3.1, Y is regular. So $M \leq G$ \square

From now on, we suppose Γ is a connected edge-transitive tetravalent metacirculant of odd order.

By the result of [18], there is no 4-arc transitive graph of valency at least 3 on odd number of vertices. The following proposition characterizes the vertex stabilisers of connected s -transitive tetravalent graphs (see [18, Theorem 1.1], [20, Lemma 2.5], and [44, Proposition 2.16])

stabilizer

Proposition 3.3. *Let Γ be a connected (X, s) -transitive tetravalent graph of odd order, and let X_α be the stabilizer of a vertex $\alpha \in V$ in X . Then one of the following holds:*

- 1) $s = 1$, X_α is a 2-group;
- 2) $s = 2$, $A_4 \leq X_\alpha \leq S_4$;
- 3) $s = 3$, $A_4 \times \mathbb{Z}_3 \triangleleft X_\alpha \triangleleft S_4 \times S_3$.

The following lemma characterises tetravalent edge-transitive metacirculants which has a quasiprimitive automorphism group.

edge-transitive

Lemma 3.4. *Let Γ be a connected tetravalent X -edge-transitive metacirculant of odd order. Then Γ is a split metacirculant of G . Suppose that X is quasiprimitive on V and contains G . Then X is almost simple.*

Proof. By Lemma 3.2, there is a metacyclic group G such that $\Gamma = \text{Cay}(G, S)$.

Since X is quasiprimitive on V . Let M be a minimal normal subgroup of X . Then $M = T^l$ where T is a simple group. Suppose X is affine. Then $M = \mathbb{Z}_p^d$ is transitive on V , and M is of odd order. By Lemma 3.2, $M \leq G$, so $d \leq 2$ as G is metacyclic. Since M is transitive on V , we have $G = M$, and G is normal in X . Thus $X = G:\text{Aut}(G, S)$. Hence $\text{Aut}(G, S) \leq S_4$, as $\text{Aut}(G, S)$ acts on S faithfully. So $X_\alpha \leq D_8$ as neither A_4 nor S_4 has a faithful representation of degree 2.

Now suppose T is insoluble. We claim that $l = 1$ and X is almost simple. First of all, we notice that X_α is a $\{2, 3\}$ -group, so $|X|_{\{2, 3\}'} = |G|_{\{2, 3\}'}$. Since G is soluble, a Hall $\{2, 3\}'$ -subgroup $G_{\{2, 3\}'}$ exists and $G_{\{2, 3\}'} < G < X$, so $G_{\{2, 3\}'}$ is a Hall $\{2, 3\}'$ -subgroup of X . Thus $X_{\{2, 3\}'}$ is metacyclic.

Suppose $l > 2$. Let p be a prime divisor of $|T|$ such that $p \neq 2, 3$. Then a Sylow p -subgroup of T^l is isomorphic to $T_p^l \leq X_p$, and X_p is a metacyclic group which is not possible.

Suppose $l = 2$. Suppose $3 \nmid |X_\alpha|$. Then X_α is a 2-group. Thus $G_{2'}$ is a Hall $2'$ -group of X . By Lemma 2.3, T has a Hall $2'$ -subgroup $T_{\{2'\}}$. Hence $T \cong \text{PSL}(2, p)$ with $T_{2'} = \mathbb{Z}_p : \mathbb{Z}_{p-1}$ by Lemma 2.5. On the other hand, $T_{2'}^2 \leq X_{2'} \cong G$. By Lemma 2.4, $T_{2'}$ is a cyclic group. So $p = 3$, and $T = A_4$ is soluble which contradicts to T is insoluble. Suppose $3 \mid |X_\alpha|$. By Lemma 3.3, $|X_\alpha|_2 \leq 16$, so $|T_2| \leq 4$ as

$(T^2)_2 \leq X_\alpha$. Check the order of non-abelian finite simple group, we have $T = \text{PSL}(2, q)$ with $q \equiv \pm 3 \pmod{8}$ or $q = 2^2$. Moreover, $T_{\{2,3\}'}^2 \lesssim X_{\{2,3\}'} \cong G$. So $T_{\{2,3\}'}$ is a cyclic group.

The order of $\text{PSL}(2, q)$ is $q(q^2 - 1)/d = q(q + 1)(q - 1)/d$, where $d = (2, q - 1)$. Suppose $p \geq 5$. Then the order of cyclic subgroup of $\text{PSL}(2, q)$ is at most $(q + 1)/2$ or p , if $q = p^f$ with $f \geq 2$, or $q = p$, respectively, where p is a prime. Suppose 3 divides q . Then 2, 3 do not divide $(q + 1)(q - 1)/8$ as $q \equiv \pm 3 \pmod{8}$. Thus $(q + 1)(q - 1)/8 = |T|'_{\{2,3\}}$. Hence there is a cyclic subgroup of order $(q + 1)(q - 1)/8$, so $(q + 1)(q - 1)/8 \leq (q + 1)/2$ or q . Since $3 \mid q$ and $q = p^f > 5$, we have the latter is not possible. So $(q + 1)(q - 1)/8 \leq (q + 1)/2$, thus $q \leq 5$ and $q = 5$ contradicts to $3 \mid q$. Suppose $3 \mid p + 1$. Similarly, 2, 3 do not divide $q(q - 1)/8$. It follows that either $q(q - 1)/8 \leq q + 1/2$ where $q = p^f$ with $f \geq 2$, or $q(q - 1)/8 \leq q$ where $q = p$. Thus $q \leq 7$, while $3 \mid q + 1$ so $q = 5$. Suppose $3 \mid q - 1$. Using the same argument, we have $q = 7$. Notice that $\text{PSL}(2, 4) \cong \text{PSL}(2, 5)$, we have T is $\text{PSL}(2, 5) \cong A_5$, or $\text{PSL}(2, 7)$. For the former, $M = A_5^2$, $X \leq M.\mathbb{Z}_2^2.\mathbb{Z}_2$ as X is quasiprimitive. Hence the maximal metacyclic subgroup of odd order in X is \mathbb{Z}_5^2 . So $|V| = 5^2$. On the other hand, $M_\alpha \triangleleft X_\alpha$ such that X_α satisfies Lemma 3.3. Since the maximal subgroup of M which is normal in X_α is $A_4 \times \mathbb{Z}_3$, we have $5^2 = |V| = |M:M_\alpha| = 4 \times 5^2$ which is not possible. For the latter, $M = \text{PSL}(2, 7)^2$. Using the same argument we can show that this is not possible too. Hence $l = 1$ and X is almost simple. \square

Quasiprimitive permutation groups which contain a transitive metacyclic subgroup are determined in [22]. Then we have the following result.

quasipri-gps

Lemma 3.5. *Let H be an almost simple quasiprimitive permutation group which contains a transitive metacyclic subgroup R of odd order. Then either $(H, H_\alpha) = (A_n, A_{n-1})$ or (S_n, S_{n-1}) with n odd, or the triple (H, R, H_α) lies in Table 2, where p is a prime, $G(q^n)$ is a transitive subgroup of $\Gamma\text{L}(1, q^n) = \mathbb{Z}_{q^n-1}:\mathbb{Z}_n$, and P_1 is a parabolic subgroup.*

row	H	R	H_α	conditions
1	$A_{p.o}$	$p:\frac{p-1}{2}$	$S_{p-2} \times o$	$o \leq 2$
2	$\text{PSL}(n, q).o$	$G(q^n).o_1$	$P_1.o_2$	$q = p^f, o_1 o_2 \cong o \leq f.(n, q - 1)$
3	$\text{PSL}(2, p).o$	$p:\frac{p-1}{2}$	$D_{(p+1)o}$	$o \leq 2, p \equiv 3 \pmod{4}$
4	$\text{PSL}(2, 11).o$	11:5	$A_4.o$	$o \leq 2$
5	$\text{PSL}(2, 11)$	11	A_5	
6	$\text{PSL}(2, 29)$	29:7	A_5	
7	$\text{PSL}(2, p)$	$p:\frac{p-1}{2}$	A_5	$p = 11, 19, 59$
8	$\text{PSL}(2, 23)$	23:11	S_4	
9	$\text{PSL}(5, 2)$	31:5	$2^6 : (S_3 \times \text{PSL}(3, 2))$	
10	$\text{PSU}(3, 8).3^2.o$	$3 \times 19:9$	$(2^{3+6}:63:3).o$	$o \leq 2$
11	$\text{PSU}(4, 2).o$	9:3	$2^4:A_5.o$	$o \leq 4$
12	M_{11}	11, 11:5	$M_{10}, M_9.2$	
13	M_{23}	23, 23:11	$M_{22}, M_{21}.2, 2^4.A_7$	

Table 2: quasiprimitive permutation groups of type AS

AS **Lemma 3.6.** *Suppose $\Gamma = \text{Cay}(G, S)$ is a connected X -edge-transitive tetravalent graph of odd order such that $G \leq X$ is quasiprimitive group of almost simple. Then (X, G, X_α) lies in Table 3.*

Proof. First of all, since X_{α} satisfies Lemma 3.3 and G is of odd order, by checking the candidates in Lemma 3.5, we have (X, X_α) can only appear as rows 3, 4, 8 or (A_5, A_4) , (S_5, S_4) . Suppose $(X, X_\alpha) = (\text{PSL}(2, p).o, D_{p+1}.o)$. Then by Lemma 3.3, $D_{p+1}.o$ is a 2-group. So X is primitive except $p = 7$. Since X are primitive for all the other five cases, we have either X is primitive, or $X = \text{PSL}(2, 7)$ and $X_\alpha = D_8$. Suppose X is primitive. Then by [18, Theorem 1.5], (X, X_1, G) is listed in rows 1, 3 and 4 of Table 1 and for each G there is only one vertex-primitive arc-transitive graph of valency 4. Suppose $X = \text{PSL}(2, 7)$ with $X_\alpha = D_8$. Then $X_{\alpha\beta} = \mathbb{Z}_2$, and $\text{Aut}\Gamma = \text{PGL}(2, 7)$ which is primitive too. So this gives the candidate of row 2 of Table 1. \square

normal **Lemma 3.7.** *Let $\Gamma = \text{Cay}(G, S)$ be a connected X -edge-transitive tetravalent graph of odd order. Suppose $G \leq X$ has a non-trivial intransitive normal subgroup M . Then G is normal in X , and $X_\alpha \leq D_8$.*

Proof. Let M be a minimal normal subgroup of X which is intransitive. Let \mathcal{B} be the set of M -orbits. Let K be the kernel of X acting on \mathcal{B} .

Case 1. Suppose $K_\alpha \neq 1$. We claim that $G \triangleleft X$. Suppose G is not normal in X . By [21, Lemma 5.6], $G = \mathbb{Z}_p^2 : \mathbb{Z}_m$ with $m \geq 3$ and $X = \mathbb{Z}_p^2 : (\mathbb{Z}_2^l : \mathbb{Z}_m)$ or $\mathbb{Z}_p^2 : (\mathbb{Z}_2^l : \mathbb{Z}_m) : \mathbb{Z}_2$ such that \mathbb{Z}_p^2 is the only minimal normal subgroup of X , where $l \geq 2$. It follows that $\mathbb{Z}_2^l : \mathbb{Z}_m$ has a faithful representation on \mathbb{Z}_p^2 , which is not possible. Thus $l = 1$ and $K_\alpha \cong \mathbb{Z}_2$. So $K = M : \mathbb{Z}_2$. By Lemma 3.2, $M < G$. Let $\overline{G} := GK/K \cong G/K \cap G$, let $\overline{X} = X/K$. We have $\overline{G} \leq \overline{X}$ is transitive on Γ_M . Since G is of odd order, Γ_M is a cycle, we have \overline{G} is regular on $V\Gamma_M$, so $\overline{G} \triangleleft \overline{X}$. Then $K.\overline{G} \triangleleft X$. Let $L = K.\overline{G} = K.(G/G \cap K)$. Since $K = (K \cap G) : \mathbb{Z}_2$, we have $L = 2|V|$ is soluble. So $G = L_{2'}$ has index 2 in L . Thus $G \text{ char } L \triangleleft X$. Hence $G \triangleleft X$.

Case 2. Suppose $K_\alpha = 1$. We will show that $G \triangleleft X$. Suppose for each normal subgroup N of X , Γ is a normal cover of Γ_N . Then let N be a maximal normal subgroup of X which is intransitive. It follows that X/N is quasiprimitive on $V\Gamma_N$. Thus by Lemma 3.2, we have $N < G$. Suppose further N is a minimal normal subgroup of X . Then $N = \mathbb{Z}_p^d$ with $d \leq 2$ as $N < G$. Moreover, X/N is quasiprimitive on Γ_N . So either X/N is an affine group such that $G/N \triangleleft X/N$ or X/N is listed in Table 1. For the former, since $G/N \triangleleft X/N$, we have $G \triangleleft X$. For the latter, $N = \mathbb{Z}_p^d$ with $d \leq 2$ and p an odd prime. By [6], the groups listed in Table 1 are not contained in $\text{Aut}(N) = \text{GL}(2, p)$ or $\text{GL}(1, p)$. Further, as p is an odd prime, and the Schur Multipliers of A_5 , $\text{PSL}(2, 7)$, $\text{PSL}(2, 11)$ and $\text{PSL}(2, 23)$ are 2, we have $X = N \times (X/N)$. Hence X has a normal subgroup H which is isomorphic to the socle of X/N . Further, H is intransitive on V . So H has at least three orbits on V , but $3 \mid |H_\alpha|$ which is not possible. Suppose N is not minimal normal subgroup of X . Let $N' \triangleleft N$ be a second maximal normal subgroup of X which is intransitive. Then N/N' is a minimal normal subgroup of X . Then Γ is a normal cover of $\Gamma_{N'}$. Let $\overline{N} := N/N'$, and $\overline{X} := X/N'$. Then $\Gamma_{N'}$ is \overline{X} -edge-transitive with $\overline{G} := G/N' \leq X/N'$ is regular on $V\Gamma_{N'}$, and $\Gamma_{N'}$ is a normal cover of Γ_N . Moreover,

$\overline{X}/\overline{N} \cong X/N$ is quasiprimitive on $V\Gamma_N$. By the same argument as above, we have this is not possible.

Now suppose there is a normal subgroup L of X , such that Γ is not a normal cover of Γ_L . Since Γ is a normal cover of Γ_M , we may suppose there are normal subgroups N, L of X such that Γ is a normal cover of Γ_N , $L > N$ such that L/N is a minimal normal subgroup of X/N and Γ is not a normal cover of Γ_L . By Lemma 3.2, $N < G$. Now $\overline{X} := X/N$ is edge-transitive on Γ_N and $\overline{G} := G/N \leq X/N$ is regular on $V\Gamma_N$. Further $\overline{L} := L/N$ is a minimal normal subgroup of $\overline{X} = X/N$ and Γ_N is not a normal cover of Γ_L . By Case 1. we have $G/N \triangleleft X/N$, so $G \triangleleft X$.

Thus $X = G:\text{Aut}(G, S)$. Since $\text{Aut}(G, S)$ acts on S faithfully, we have $\text{Aut}(G, S) \leq S_4$. So $X_\alpha \leq D_8$ as neither A_4 nor S_4 has a representation of degree 2. \square

Now we study normal edge-transitive metacirculants.

normal-meta

Lemma 3.8. *Let $\Gamma = \text{Cay}(G, S)$ be a connected X -edge-transitive metacirculants of odd order such that $G \triangleleft X$. Then one of the following holds:*

- 1) Γ is an X -half-transitive metacirculant of non-abelian metacyclic group, and $X = G:\mathbb{Z}_2$;
- 2) Γ is an X -arc-transitive metacirculant of an abelian metacyclic group, and $X = G:X_\alpha$, where $\mathbb{Z}_2 < X_\alpha \leq D_8$.

Proof. Let $\Gamma = \text{Cay}(G, S)$, where $S = \{x, x^{-1}, y, y^{-1}\}$ is a generating subset of G . By Lemma 3.7, $X \leq G:\text{Aut}(G, S)$ with $\text{Aut}(G, S)$. Let $\sigma \in X_\alpha$ where $\alpha \in V$. Suppose $x^\sigma = x^{-1}$, $y^\sigma = y^{-1}$. Then $o(\sigma) = 2$. Suppose further σ fixes $z = x^i y^j$ for some integers i, j . Then

$$x^i y^j = (x^i y^j)^\sigma = x^{-i} y^{-j},$$

and so $x^{2i} = y^{-2j}$. Since $o(x) = o(y)$ are odd, we have $x^i = y^{-j}$. Then $z = x^i y^j = 1$, that is, σ fixes no non-identity of G . By [?, Exercise 1.50], G is abelian. Since all edge-transitive Cayley graphs of abelian groups are arc-transitive, we have $\text{Aut}(G, S) > \mathbb{Z}_2$. Further, by Lemma 3.4, $\text{Aut}(G, S) = D_8$.

Suppose G is non-abelian. Suppose there is an element $\tau \in \text{Aut}(G, S)$ such that $x^\tau = x^{-1}$, or $y^\tau = y^{-1}$. Without loss of generality, we suppose $x^\tau = x^{-1}$. Since G is non-abelian, $y^\tau \neq y^{-1}$. So $y^\tau = y$. As Γ is edge-transitive, $\text{Aut}(G, S)$ has at most two orbits on S . So there is another element say $\gamma \in \text{Aut}(G, S)$ such that $y^\gamma = y^{-1}$, and $x^\gamma = x$. Thus $x^{\gamma\tau} = x^{-1}$, $y^{\gamma\tau} = y^{-1}$, and $o(\gamma\tau) = 2$ which contradicts to G is non-abelian. Without loss of generality, we suppose $x^\tau = y$ and suppose further that $y^\tau \neq x$. Then $y^\tau = x^{-1}$. Hence $x^{\tau^2} = x^{-1}$ which is not possible too. So $y^\tau = x$. Thus $\text{Aut}(G, S) \cong \mathbb{Z}_2$. \square

4. EDGE-TRANSITIVE TETRAVALENT SYLOW-CIRCULANTS

As an application of Lemma 3.8, in this section we study Sylow-circulants of odd order and determine the number of non-isomorphic edge-transitive tetravalent Sylow-circulants.

Recall that a group is called Sylow-cyclic, if all Sylow subgroups of it are cyclic. Let $G = H \times K = \langle c \rangle \times \langle (a, b) \rangle = \langle a \rangle : \langle bc \rangle$ be a Sylow-cyclic group, where $K = \langle a, b \mid a^m = b^n = 1, a^b = a^r \rangle$, $r \not\equiv 1 \pmod{m}$, $r^n \equiv 1 \pmod{m}$, such that no Sylow- p subgroup of $\langle b \rangle$ lies in the centre of G . We first study the edge-transitive tetravalent

Cayley graph of G such that $H = 1$. By Lemma [3.8](#), we know all edge-transitive tetravalent graphs $\text{Cay}(G, S)$ are normal Cayley graphs. So $\text{Aut}\Gamma = G:\text{Aut}(G, S)$. Thus to get a better understand of the automorphism groups and these graphs, we will study the automorphism group $\text{Aut}(G)$ of G , then determine $\text{Aut}(G, S)$.

Remark: Let $G = \mathbb{Z}_m:\mathbb{Z}_n$ be a Sylow-cyclic metacyclic group. Then $(m, n) = 1$.

Now $G = \langle a, b \mid a^m = b^n = 1, a^b = a^r \rangle$ is a non-abelian metacyclic group. Then $r \not\equiv 1 \pmod{m}$ and $r^n \equiv 1 \pmod{m}$. By calculating, the following equations hold.

$$a^u b^v = b^v a^{r^v u} \quad (\text{eqn:1})$$

$$(a^{u_1} b^{v_1})(a^{u_2} b^{v_2}) = b^{v_1+v_2} a^{u_1 r^{v_1+v_2} + u_2 r^{v_2}} \quad (\text{eqn:2})$$

$$(a^u b^v)^k = b^{kv} a^{u(r^{kv} + r^{(k-1)v} + \dots + r^v)} = b^{kv} a^{ur^v \frac{r^{vk} - 1}{r^v - 1}} \quad (\text{eqn:3})$$

order

Lemma 4.1. *Let $G = \langle a, b \mid a^m = b^n = 1, a^b = a^r \rangle$ be a non-abelian Sylow-cyclic group. Suppose further that G is not a product of its any two non-trivial subgroups. Then $o(a^i b^j) = o(b^j) = n$ for $0 \leq i \leq m$ and $(j, n) = 1$.*

Proof. We first prove that $(r-1, m) = 1$ and $\langle a \rangle \cap \mathbf{Z}(G) = 1$. Let p^α be the largest p -power divisor of $(r-1, m)$. Then $r \equiv p^\alpha x + 1 \pmod{m}$ where $(x, p) = 1$. Since $r^n \equiv 1 \pmod{m}$, we have $(r^n - 1) \equiv (p^\alpha x)^n + n(p^\alpha x)^{n-1} + \dots + n(p^\alpha x) \pmod{m}$. Thus p^α is the largest prime divisor of m as $(n, m) = 1$. It follows that if $\alpha \neq 0$, then $(a^{m/p^\alpha})^b = a^{rm/p^\alpha} = a^{m/p^\alpha}$, that is $\langle a \rangle_p \subset \mathbf{Z}(G)$. So $G = \langle a \rangle_p \times G'$ which contradicts to our assumption. So $\alpha = 1$ and $(r-1, m) = 1$. Suppose $a^l \in \mathbf{Z}(G)$. Then $(a^l)^b = a^{rl} = a^l$. Hence $l \equiv 0 \pmod{m}$ as $(r-1, m) = 1$. So $\langle a \rangle \cap \mathbf{Z}(G) = 1$. Moreover, $(a^i b)^n = b^n a^{ir \frac{r^n - 1}{r - 1}} = a^{ir \frac{r^n - 1}{r - 1}}$. Since $(r-1, m) = 1$, we have $a^{ir \frac{r^n - 1}{r - 1}} = 1$, that is $(a^i b)^n = 1$. So $o(a^i b) = n$ for any i .

Now suppose $p \mid (r^j - 1, m)$. Then $(a^{m/p})^{b^j} = a^{r^j m/p} = a^{m/p}$. Since $(j, n) = 1$, we have $a^{m/p} \in \mathbf{Z}(G)$ which is not possible. So $(r^j - 1, m) = 1$ and then $o(a^i b^j) = n$. \square

The following lemma give some properties of $\text{Aut}(G)$.

auto-G

Lemma 4.2. *Let $G = \langle a, b \mid a^m = b^n = 1, a^b = a^r \rangle$ such that $r \neq 1$ and $r^n \equiv 1 \pmod{m}$. Let n_0 be the smallest integer such that $r^{n_0} \equiv 1 \pmod{m}$. Suppose further that G is not a product of its any two non-trivial subgroups. Then $\sigma \in \text{Aut}(G)$ if and only if:*

$$\begin{aligned} \sigma : \quad a &\rightarrow a^s & \text{where } 1 \leq s < m, (m, s) = 1 \\ b &\rightarrow a^t b^{1+ln_0} & \text{where } 0 \leq l < n/n_0, 1 \leq t \leq m. \end{aligned}$$

In particular, b^j is conjugate to $a^t b^{j+ln_0}$ under $\text{Aut}(G)$ for $(j, n) = 1$, and b^i not conjugate to b^{-i} under $\text{Aut}(G)$ for any $b^i \notin \mathbf{Z}(G)$.

Proof. Let σ be an automorphism of G . By assumption, $\langle a \rangle \text{ char } G$, so $a^\sigma = a^s$ for some integer s such that $o(a^s) = m$. Hence $(s, m) = 1$. Now suppose $b^\sigma = a^t b^u$. Then

$$\begin{aligned} (ab)^\sigma &= a^\sigma b^\sigma = a^s a^t b^u = a^t a^s b^u \\ &= (ba^r)^\sigma = a^t b^u a^{rs}. \end{aligned}$$

Hence $a^s b^u = b^u a^{rs}$, that is, $b^{-u} a^s b^u = a^{rs}$. So $b^{-(u-1)} a^s b^{u-1} = b(b^{-u} a^s b^u) b^{-1} = b a^{rs} b^{-1} = a^s$. Thus $b^{u-1} \in \mathbf{Z}(G)$, as $(s, m) = 1$. Since n_0 is the smallest integer such

that $r^{n_0} \equiv 1 \pmod{m}$, we have $\mathbf{Z}(G) \cap \langle b \rangle = \langle b^{n_0} \rangle$. Thus $u = 1 + ln_0$ for some integer $0 \leq l < n/n_0$.

On the other hand, Let σ be defined as in the lemma. Since $G = \langle a^s, a^t b^{1+ln_0} \mid o(a^s) = m, o(a^t b^{1+ln_0}) = n, (a^s)^{a^t b^{1+ln_0}} = (a^s)^r \rangle$, we have σ is an automorphism of G . Similarly, suppose $a^\tau = a^s$, where $(s, m) = 1$, and $(b^j)^\tau = a^t b^{j+ln_0}$. Then $G = \langle a^s, a^t b^{j+ln_0} \mid o(a^s) = m, o(a^t b^{j+ln_0}) = n, (a^s)^{a^t b^{j+ln_0}} = (a^s)^{r^j} \rangle$. Then τ is an automorphism of G , so b^j is conjugate to $a^t b^{j+ln_0}$ under $\text{Aut}(G)$.

Suppose $(b^i)^\sigma = b^{-i}$. Then $b^{i+ln_0} = b^{-i}$. Thus $2i + ln_0 \equiv 0 \pmod{n}$. Since $n_0 \mid n$, we have $n_0 \mid i$, and $b^i \in \mathbf{Z}(G)$. Thus b^i is not conjugate to b^{-i} under $\text{Aut}(G)$ for any $b^i \notin \mathbf{Z}(G)$. So the lemma holds. \square

Let $\Gamma = \text{Cay}(G, S)$ be a connected edge-transitive metacirculants with $S = \{x, y, x^{-1}, y^{-1}\}$. By Lemma 3.8, we have $\text{Aut}\Gamma = G:\text{Aut}(G, S)$ where $\text{Aut}(G, S) = \mathbb{Z}_2$. Thus the involutions in $\text{Aut}(G)$ play an important role in study edge-transitive metacirculants. Then following lemma determines the involution automorphisms of G .

Let \mathbf{Z} be the centre of G . Let n_0 be the smallest integer such that $r^{n_0} \equiv 1 \pmod{m}$. Then $a^{b^{n_0}} = a^{r^{n_0}} = a$. So $b^{n_0} \in \mathbf{Z}$.

involution-auto

Lemma 4.3. *Let σ be an involution automorphism of G . Then*

$$\begin{aligned} \sigma : b &\rightarrow a^i b & \text{where } o(a^i b) = n \\ a &\rightarrow a^s & \text{where } s^2 \equiv 1 \pmod{m}. \end{aligned}$$

Proof. Suppose $b^\sigma = a^i b^{1+ln_0}$, $a^\sigma = a^s$. Then

$$b^{\sigma^2} = (a^i b^{1+ln_0})^\sigma = a^{is} (a^i b^{1+ln_0})^{(1+ln_0)} = b^{(1+ln_0)^2} a^x, \text{ as } [a, b] \in \langle a \rangle.$$

Since $o(\sigma) = 2$, $b^{(1+ln_0)^2} = b$, that is, $(1 + ln_0)^2 \equiv 1 \pmod{n}$. So $ln_0(ln_0 + 2) \equiv 0 \pmod{n}$. We know that $\langle b^{n_0} \rangle \leq \mathbf{Z}$. Let p be a prime divisor of $o(b^{n_0})$ and suppose $(p, n_0) = 1$. Let P be a Sylow- p subgroup of $\langle b^{n_0} \rangle$. Since, $(p, n_0) = 1$, we have P is a Sylow- p subgroup of $\langle b \rangle$. Hence there is a Hall- p' subgroup $B_{p'}$ of $\langle b \rangle$ such that $\langle b \rangle = P:B_{p'}$. While $P \leq \mathbf{Z}$, we $\langle b \rangle = P \times B_{p'}$. It follows that $G = P \times (\langle a \rangle : B_{p'})$ which contradicts to our assumption. Thus p divides n_0 , so $(p, ln_0 + 2) = 1$. Hence $(ln_0 + 2, n) = 1$, and $ln_0 \equiv 0 \pmod{n}$. \square

abj in S

Lemma 4.4. *S has the form $\{a^{i_1} b^j, a^{i_2} b^j, (a^{i_1} b^j)^{-1}, (a^{i_2} b^j)^{-1}\}$ for some $1 \leq i_1 < i_2 \leq m$, where $(j, n) = 1$.*

Proof. Since $\langle S \rangle = G$, $\langle \bar{S} \rangle = \bar{G} = G/\langle a \rangle = \langle \bar{b} \rangle \cong \mathbb{Z}_n$ where \bar{S} , \bar{b} are the image of S , b under the map from G to $G/\langle a \rangle$, respectively. Thus there is an element, say x , with the form $a^{i_1} b^j$ where $(j, n) = 1$.

Further, by Lemma 3.8, $\text{Aut}(G, S) = \mathbb{Z}_2$. Let σ be an involution in $\text{Aut}(G, S)$. By Lemma 4.2, σ can not maps x to x^{-1} . Thus σ map x to y (or y^{-1}). By Lemma 4.3, $y = a^{i_2} b^j$. \square

The following lemma gives a sufficient and necessary condition of $\langle S \rangle = G$.

generators

Lemma 4.5. *Let $G = \langle a \rangle : \langle b \rangle$ as defined above, and let $S = \{a^{i_1} b^j, a^{i_2} b^j\}$, $1 \leq i_1, i_2 \leq m$, $1 \leq j \leq n$. Then $\langle S \rangle = G$ if and only if $(j, n) = 1$, $(i_2 - i_1, i_1[r]_n \pmod{m}) = 1$.*

Proof. Suppose $(j, n) = 1$ and $(i_2 - i_1, i_1[r]_n) = 1$. Then $a^{i_2-i_1} = a^{i_2}b^j \cdot (a^{i_1}b^j)^{-1} \in \langle S \rangle$, and $(a^{i_1}b^j)^n = a^{i_1[r]_n} = a^{i_1[r]_n} \in S$. Since $(i_2-i_1, i_1[r]_n) = 1$, we have $\langle a \rangle \subset \langle S \rangle$. Then $b^j \in \langle S \rangle$. Hence $\langle b \rangle \subset \langle S \rangle$, as $(j, n) = 1$.

Now suppose $\langle S \rangle = G$. Let ϕ be the natural map from G to $G/\langle a \rangle$. Let \bar{S}, \bar{b} be the image of S under ϕ . Then $\langle \bar{S} \rangle = \langle \bar{b}^j \rangle$. Since $\langle S \rangle = G$, $\langle \bar{S} \rangle = \langle \bar{b}^j \rangle = \bar{G} \cong \mathbb{Z}_n$. Hence $(j, n) = 1$.

Now $G = \langle S \rangle = \langle a^{i_1}b^j, a^{i_2-i_1} \rangle$. For convenience, denote $i = i_2 - i_1$. Then $\langle a^i \rangle \triangleleft \langle S \rangle$. Thus every elements in G can be written as $a^{is}(a^{i_1}b^j)^t$, where s, t are integers. We may suppose $a = a^{is'}(a^{i_1}b^j)^{nt'}$. For convenience, let $x = (a^{i_1}b^j)^n = a^{i_1[r]_n} = a^l$. Then $a = a^{is'+lt'}$. So $(i, i_1[r]_n \pmod{m}) = (i, i_1[r]_n \pmod{m}) = 1$, that is, $(i_2 - i_1, i_1[r]_n \pmod{m}) = 1$. \square

By Lemma [4.4](#), every edge-transitive tetravalent Cayley graph of non-abelian metacyclic group is isomorphic to $\text{Cay}(G, S)$, where $S = \{a^{i_1}b^j, a^{i_2}b^j, (a^{i_1}b^j)^{-1}, (a^{i_2}b^j)^{-1}\}$. Moreover, as $\text{Aut} \Gamma = G:\mathbb{Z}_2$, $(|G|, |\text{Aut} \Gamma_\alpha|) = 1$. By Proposition [2.2](#), Γ is a CI-graph.

In the following, we will study the isomorphism between edge-transitive metacirculants. Thus it is sufficient to determine the conjugation of different generating set S .

-classification

Lemma 4.6. *Let Γ be a connected normal edge-transitive tetravalent graph. Then $\Gamma \cong \text{Cay}(G, S_j)$ with $S_j = \{b^j, ab^j, b^{-j}, (ab^j)^{-1}\}$, where $(j, n) = 1$ and $1 \leq j < n_0$.*

Proof. Let $\Gamma = \text{Cay}(G, S)$. Since $\langle S \rangle = G$, $\langle \bar{S} \rangle = \bar{G} = G/\langle a \rangle = \langle \bar{b} \rangle \cong \mathbb{Z}_n$ where \bar{S}, \bar{b} are the image of S, b under the map from G to $G/\langle a \rangle$, respectively. Thus there is an element, say x , with the form $a^{i_1}b^j$ where $(j, n) = 1$. By Lemma [4.2](#), a^{tj+ln_0} is conjugated to b^j under $\text{Aut}(G)$. Thus S is conjugate to S'_j , where $b^j \in S'_j$. Hence $\Gamma \cong \text{Cay}(G, S'_j)$. Further, by Lemma [3.8](#), $\text{Aut}(G, S'_j) = \mathbb{Z}_2$. Let σ be an involution in $\text{Aut}(G, S'_j)$. By Lemma [4.3](#), suppose σ map b^j to $a^i b^j$. Then $S'_j = \{b^j, a^i b^j, b^{-j}, (a^i b^j)^{-1}\}$. It follows that $(i, m) = 1$, as $\langle S'_j \rangle = G$. Since $(i, m) = 1$, we have there is an integer s such that $is \equiv 1 \pmod{m}$. It follows that $(s, m) = 1$. Moreover, $(t_2, m) = 1$, so $(st_2, m) = 1$. Let

$$\tau : b^j \rightarrow ab^j, a \rightarrow a^{-s}.$$

Then $S_j'^\tau = S_j = \{b^j, ab^j, (b^j)^{-1}, (ab^j)^{-1}\}$. Hence $\Gamma_1 \cong \Gamma_2$. Thus $\Gamma \cong \text{Cay}(G, S_j)$ where $S_j = \{b^j, ab^j, b^{-j}, (ab^j)^{-1}\}$. The lemma holds. \square

Proof of Theorem [1.2](#): Let $G = H \times K = \langle c \rangle \times (\langle a, b \rangle) = \langle a \rangle : \langle bc \rangle$, where $K = \langle a, b \mid a^m = b^n = 1, a^b = a^r \rangle$, $r^n \equiv 1 \pmod{m}$, such that no Sylow- p subgroup of $\langle b \rangle$ lies in the centre of G . Suppose first that $H = 1$. Then by Lemma [3.4](#), the theorem holds.

Now we suppose $H \neq 1$. Since G is a metacyclic group, we have $(o(c), n) = 1$. Thus $H \text{ char } G$ and $K \text{ char } G$. Let $S = \{x, y, x^{-1}, y^{-1}\}$, where $x = x_1 x_2$ and $y = y_1 y_2$ with $x_1, y_1 \in H, x_2, y_2 \in K$.

As Γ is edge-transitive, by Theorem [1.1](#), there is an involution automorphism σ such that $x^\sigma = y$. Thus $x_i^\sigma = y_i, i = 1, 2$. Hence $x_1 \neq 1$ as $\langle S \rangle = G$, and $\text{Cay}(K, S_2)$ is a tetravalent edge-transitive metacirculant where $S_2 = \{x_2, y_2, x_2^{-1}, y_2^{-1}\}$. Thus by Lemma [3.4](#), S_2 satisfies Lemma [3.4](#). Now consider x_1, y_1 , as $\langle S \rangle = G$, we have

$\langle x_1, y_1 \rangle = H$. Suppose $x_1 = c^s$. Then $y_1 = c^{st}$ with $t^2 \equiv 1 \pmod{l}$. Thus $(s, l) = 1$ as $\langle x_1, y_1 \rangle = H$. We may suppose $\{x_1, y_1\} = \{c, c^t\}$ as c is conjugate to c^t , where $(s, l) = 1$. Then $\{x_1, y_1\}^\sigma = \{c, c^{-1}\}$, where $c^\sigma = c^t$. Thus by Proposition 2.2, the Theorem holds. CI-criterion

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